# Coregularity of smooth Fano 3folds.

(j.w. A. Avilov and V. Przyjalkowsky) / $\mathbb{C}$

$X$ – (smooth / klt) projective variety

Fano, if $-K_X$ ample.

$|-K_X| \ni D$ - smooth ($X$ - smooth Fano, $\dim X = 3$)

∃ classification of smooth Fano 3folds (Iskovskikh – Mori – Mukai)

$|-lK_X|$, $l \geq 1$.

$\alpha$ - invariant, $\delta$ - invariant, complexity, coregularity

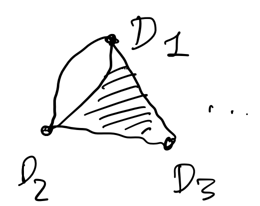
$|-K_X| \ni D = \sum_{i=1}^{N} D_i$. Estimate $N$. (Mirror symmetry)

$|-lK_X|$.

---

**Def**: $X$ - smooth proj, $D = \sum_{i=1}^{N} D_i$ - snc

$(X, D) \longrightarrow \mathcal{D}(D)$ - dual complex of $D$.




irr. comp of $D_{i_1} \cap \ldots \cap D_{i_k} \longleftrightarrow \Delta^{k-1}$

irr. comp. of $D_{i_1} \cap \ldots \cap \widehat{D}_{i_j} \cap \ldots \cap D_{i_k} \longleftrightarrow \Delta^{k-2}$

---

**Def**: $(X, D)$ – (sub-)pair, $D = \sum a_i D_i$, $0 \leq a_i \leq 1$, $a_i \in \mathbb{Q}$ ($-\infty \leq a_i \leq 1$).

normal proj. $K_X + D$ – $\mathbb{Q}$-Cartier

**Def**: $(X, D)$ – (sub-)pair.

$f: Y \xrightarrow[\text{birational}]{\text{proper}} X$ – log resolution

$Y$ – smooth, $f_*^{-1} D \cup \mathrm{Exc}(f)$ – snc.

pair

**Def:** $(X, D)$ - lc sing-s, if. $(Y, D_Y)$ - sub-pair
$f: Y \to X$. - log resolution
$f^*(K_X + D) =: K_Y + D_Y$. $\quad D_Y \leq 1$.

**Def:** $(X, D)$ - CY pair, if $K_X + D \equiv 0$.

**Def:** $(X, D)$ - lc CY pair.
$f: Y \to X$ - log resolution
$(Y, D_Y)$ - subpair.
$\mathcal{D}(D) := \mathcal{D}(D_Y^{=1})$

**Fact** (de Fernex - Kollár - Xu) $\mathcal{D}(D)$ - does not depend on $f$.
(up to PL-homeo).

**Def** (Shokurov): $\operatorname{reg}(X, D) = \dim \mathcal{D}(D)$.

**Example:** $X$ - Fano variety, $D \in \frac{1}{e}|-eK_X|$.
If $(X, D)$ - lc $\quad \operatorname{reg}(X, D)$.

**Def** (Moraga): $X$ - (klt) Fano variety.
$$\operatorname{reg}_e(X) = \max\{\operatorname{reg}(X, D) \mid D \in \tfrac{1}{e}|-eK_X|,\ (X, D) - \text{lc}\}$$
$$\operatorname{reg}(X) = \max_{e \geq 1}\{\operatorname{reg}_e(X)\}.$$
$$\operatorname{reg}(X) \in \{-1, 0, \ldots, \dim X - 1\}$$
$-1$: exceptional Fano
$$\operatorname{coreg}_e(X) = \dim X - 1 - \operatorname{reg}_e(X)$$
$$\operatorname{coreg}(X) = \dim X - 1 - \operatorname{reg}(X).$$

---

**Example:** • $X$ - toric Fano variety.
$D = \sum D_i$ - torus-invariant
$(X, D)$ - lc, $K_X + D \sim 0$.
$\Rightarrow \operatorname{coreg}(X) = 0$.

- $X$ – smooth del Pezzo surface.

$1 \leq (-K_X)^2 = d \leq 9$.

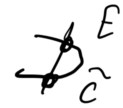
<u>Prop</u>: (a) $d \geq 2 \Rightarrow \mathrm{coreg}(X) = 0$.

(B) $d = 1 \Rightarrow$ for general $X$, $\mathrm{coreg}(X) = 0$,

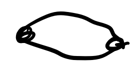
for special $X$, $\mathrm{coreg}(X) = 1$.

<u>Proof</u>: $d \geq 2$ $\quad |-K_X| \ni C$, $\quad C = $ P $\quad (X, C)$

$\mathrm{Bl}_P X =: Y \longrightarrow X \qquad (Y, D_Y)$




$D_Y = E + \tilde{C}$



$\dim \mathcal{D} = 1$

$\mathrm{coreg}(X) = 0$.

$d = 1$. For general $X$, $\exists\, |-K_X| \ni C = $ $\quad \mathrm{coreg}(X) = 0$.

$\mathbb{P}(1, 1, 2, 3) \supset X = \{ f_6(x, y) + z^3 + w^2 = 0 \}$.
$\quad\quad x \quad y \quad z \quad w$

$|-K_X|$ $\quad$ smooth $\quad \Rightarrow \mathrm{coreg}_1(X) = 1$. $\quad \mathrm{coreg}(X) = 0$.

$\mathrm{coreg}_2(X) = 1$.

<u>Thm</u> (Figueroa–Filipazzi–Moraga–Peng). $X$ – klt Fano,
$\mathrm{coreg}(X) = 0 \Rightarrow \mathrm{coreg}_1(X) = 0$ or $\mathrm{coreg}_2(X) = 0$.

$|-K_X|$, $|-2K_X|$.

---

$X$ – smooth Fano 3fold.

105 families.

<u>Thm (ALP)</u>: • For 100 families, $\mathrm{coreg}(X) = 0$
for general elements.

- For 91 families, $\mathrm{coreg}(X) = 0$ for any element.
- For 1.1, 1.2 we have $\mathrm{coreg}(X) \geq 1$ for a general element.

$X \xrightarrow{2:1} \mathbb{P}^3 \supset S_6 \qquad X_4 \subset \mathbb{P}^4$

$(-K_X)^3 \leq 10. \qquad \leq 64.$

sextic double solid     coreg = 0. !

- For 1.3 1.4 we have $\operatorname{coreg}_1(X) = 2$. for a general element.

$X_{2\cdot 3} \subset \mathbb{P}^5$     $X_{2\cdot 2\cdot 2} \subset \mathbb{P}^6$

- For 1.5 ??? we have $\operatorname{coreg}(X) \leq 1$ for a general $X$.

$Gr(2,5) \cap H_1 \cap H_2 \cap Q$.

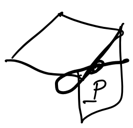
---

$i = 1, \ \rho = 1$.

To show that $\operatorname{coreg}(X) = 0$, we construct explicit boundaries.

<u>del Pezzo threefolds:</u> $i(X) = 2, \ \rho(X) = 1$.

$(X, D_1 + D_2), \quad D_i \sim \frac{-K_X}{2}$.




$\longleftarrow \mathrm{Bl}_p X =: Y$     $\operatorname{coreg}(X) = 0$.

$i \geq 2$ easy.

$i(X) = 3$.     $Q \subset \mathbb{P}^4$.

---

$\rho \geq 2$.     $(X, D) \longrightarrow (X_1, D_1)$.

$X = \mathrm{Bl}_C \mathbb{P}^3 \longrightarrow X_1 = \mathbb{P}^3 \supset C$.

---

$\rho = i = 1$.     <u>Main series.</u>

If $\underline{(-K_X)^3 \geq 12}$.     Iskovskikh double projection.

$$\begin{tikzcd}
 X' \arrow[r, dashed] \arrow[d, "\mathrm{Bl}_L"'] & X^{+} \arrow[d, "\mathrm{bir.}"] \\
 (X, D) \arrow[r, dashed, "|H-2L|"'] & Z \supset D_Z
\end{tikzcd}$$

$\operatorname{coreg}(Z, D_Z) = 0$.

$\underline{X_4} \subset \mathbb{P}^4$. [For a general quartic. $\operatorname{coreg} > 0$.

$|-K_X|$, $|-2K_X|$.

$\ni$

$D$ $\quad$ $p \in X$ $\quad$ $(\mathbb{C}^3, D)$ − strictly lc (not plt)

$D = \{f = 0\}$.

<u>Prop</u>: $(\mathbb{C}^3, D = \{f = 0\})$ − strictly lc.

$$f = \sum_{i \geq 0} f_i \Rightarrow \begin{cases} \bullet \text{ either } f_2 = 0. \\ \bullet \text{ or } f_2 = x_1^2,\ (f|_{x_1=0})_3 = 0, \\ \bullet \text{ or } f_2 = x_1^2,\ (f|_{x_1=0})_3 = x_2^3, \\ \quad (f|_{x_1 = x_2 = 0})_4 = 0. \end{cases}$$

$(f_0 = f_1 = 0)$

(similar to computation of $\underbrace{\alpha\text{-invariant}}_{\text{by Cheltsov}}$).

$\underbrace{(\mathbb{C}^3, \tfrac{1}{2}D)}$ − strictly lc

$\Rightarrow$ $(\mathbb{C}^3, D)$ − worse than lc.

---

$\underline{X_{2\cdot 3}}$, $\underline{X_{2\cdot 2\cdot 2}}$

---

<u>Corollary</u>: $(-K_X)^3 \geq 24 \Rightarrow \operatorname{coreg}(X) = 0$. for any element

$(-K_X)^3 \geq 12 \Rightarrow \operatorname{coreg}(X) = 0$ for general element.

<u>Conjecture</u>: there is some bound:

$(-K_X)^n \geq f(n)$ then $\operatorname{coreg}(X) = 0$.

---

$(-K_X)^2 \geq 2 \Rightarrow \operatorname{coreg}(X) = 0$.

---

$(X, D)$